Covariant background field calculation of ultraviolet counterterms for the nonlinear sigma model in three dimensions[*]

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Abstract

Based on the covariant background field method, we calculate the ultraviolet counterterms up to two-loop order and discuss the renormalizability of the three-dimensional nonlinear sigma models with arbitrary Riemannian manifolds as target spaces. We investigate the bosonic model and its supersymmetric extension. We show that at the one-loop level these models are renormalizable and even finite when the manifolds are Ricci-flat. However, at the two-loop order, we find non-renormalizable counterterms in all cases considered, so the renormalizability and finiteness of such models are completely lost in this order.

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arXiv:hep-th/9605198v1 28 May 1996

1. Introduction

Historically non-linear sigma models have been introduced by Schwinger[1] and originally studied in the context of current algebra[2], but later, they have been proved an excelent theoretical laboratory, since in two-dimensional space-time they are similar to four-dimensional Yang–Mills theory[3, 4]. Afterwards, they have been used, in three dimensions, to study Fermi-Boson transmutation, anyonic high-T_c superconductivity and related topics[5]. These models have also been helpful in the low-energy description of string(or superstring) theory[6]. In this late case, they are defined in two dimensions and are thus renormalizable. Indeed with the background field method[7] one is able to obtain the counterterms as functions of the gravitational fields[8, 9]. Moreover conformal invariance restraints these counterterms to zero, definning quantum corrections to Einstein equations[6, 10]. Hence, since in three dimensions they are models for membranes(or supermembranes)[11], in the same sense as the two-dimensional cases are models for strings(or superstrings), it is natural and relevant to extend these methods to three space-time dimensions. The first step in this direction is to investigate their renormalizability which as is well known may be achieved in the large n expansion for $O(n)$ and $SU(n)$ invariant models[12]. However, perturbative calculations are essential in order that one may apply the above procedure.

In the present work we deal with the question of whether non-linear sigma models in three-dimensional space-time may be perturbatively renormalizable or not. As a matter of fact, such models are nonrenormalizable in the usual Dyson power-counting sense, nevertheless it still may be possible to restore their renormalizability cancelling all divergences by including higher terms in the Lagrangian and finding a counterterm to absorb every infinity. So the main purpose of this paper is to calculate explicitly such counterterms and know what are and under which conditions generalized non-linear sigma models may be perturbative renormalizable and even finite. We carry out this program following the work of Alvarez-

Gaumé, Freedman and Mukhi[9] and making calculations for the bosonic and supersymmetric models on arbitrary manifolds, up to two-loop level. However, in our case, due to odd-dimensional nature of the space-time, the ultraviolet structure is quite different and special cares are taken throughout the calculations. We also examine some particular cases such as those models defined on Ricci-flat and symmetric spaces. We hope that the results obtained here can be useful to implement the above mentioned method of extracting low-energy information in the membrane theory context, at least in a given order.

The paper is organized as follows. Section 2 contains the useful covariant background field for the bosonic three-dimensional non-linear sigma model as well as the explicit calculations of the counterterms at the one- and two-loop level. In section 3 we present the calculations throught two-loop order for the supersymmetric extension of the previous model. In section 4 we discuss the results obtained and draw our conclusions.

2. Calculation of the counterterms for the bosonic non-linear sigma model

We shall begin studing the case of the three-dimensional purely bosonic non-linear sigma model defined by the action

$$\mathcal{S}[\phi(x)] = \frac{1}{2} \int d^3x \, g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j \tag{1}$$

where the field ϕ is a map from a three-dimensional Minkowski space-time $\mathcal{X}$(with metric tensor $\eta^{\mu\nu}$ such that $\eta^{00} = -\eta^{11} = -\eta^{22} = 1$) taken as a base space to an arbitrary Riemannian manifold $\mathcal{M}$ taken as target space and $g_{ij}(\phi(x))$ are the components of the given metric tensor on $\mathcal{M}$.

As already mentioned in the Introduction, in the usual power-counting sense, the theory defined in equation (1) is not renormalizable. Therefore, we shall investigate the ultraviolet divergences and discuss the perturbative renormalizability of this model considering terms of

higher dimensions, which are discarded in the two-dimensional case[9], calculating explicitly all the counterterms needed. In order to carry out this program we shall use here the background field method[7] explicitly covariantized via the normal coordinate expansion[13]. In fact, this procedure, which has been a powerful computational tool in Quantum Field Theory, allows us to compute radiative corrections and the effective action in a manifestly covariant way preserving the symmetries of the model under consideration[14]. For the non-linear sigma models it is already known[8, 9, 15] and consists in splitting the field ϕ^i into a classical(background) field φ^i and a quantum field π^i, taking, in followed, π^i as a function of a new covariant quantum field ξ^i in terms of which the normal coordinate is defined. Moreover, using the definitions, $\xi^i = e^i_a \xi^a$, $e^i_a e^j_a = g^{ij}$, $D^\mu \xi^a = \partial^\mu \xi^a + \omega_i^{ab} \partial^\mu \varphi^i \xi^b$, where e^i_a is a vielbein, ω_i^{ab} is the spin connection of the manifold(with Latin indices $i, j, k, \cdots$) given by

$$
\begin{aligned}
D_i e^a_j &\equiv e^a_{j;i} = \partial_i e^a_j + \omega_i^{ab}(e) e_{bj} - \Gamma^k_{ji} e^a_k = 0 \\
\omega_i^{ab} &= -e^{bj} \nabla_i e^a_j = -e^{bj} \partial_i e^a_j + e^{bj} \Gamma^k_{ij} e^a_k
\end{aligned}
\tag{2}
$$

and Γ^k_{ij} the usual Christoffel symbol, one can move to tangent space(with Latin indices $a, b, c, \cdots, h$)and get for the action (1) the following useable standard expansion(we refer to [9] for details)

$$
\mathcal{S}[\varphi + \pi] = \mathcal{S}^{(0)}[\varphi] + \mathcal{S}^{(2)}[\varphi] + \mathcal{S}^{(3)}[\varphi] + \mathcal{S}^{(4)}[\varphi] + \cdots
\tag{3}
$$

$$
\begin{aligned}
\mathcal{S}^{(2)} &= \frac{1}{2} \int d^3x \left\{ \left[R_{iabj}(\varphi) \partial_\mu \varphi^i \partial^\mu \varphi^j - A_\mu^{ac} A^{\mu cb} \right] \xi^a \xi^b + \partial_\mu \xi^a \partial^\mu \xi^a - A_\mu^{ab} \xi^a \overleftrightarrow{\partial_\mu} \xi^b \right\} \\
\mathcal{S}^{(3)} &= \frac{1}{2} \int d^3x \left\{ \left[\frac{1}{3} D_a R_{ibcj} \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{4}{3} R_{iabd} A_\mu^{dc} \partial^\mu \varphi^i \right] \xi^a \xi^b \xi^c + \frac{4}{3} R_{iabc} \partial_\mu \varphi^i \xi^a \xi^b \partial^\mu \xi^c \right\} \\
\mathcal{S}^{(4)} &= \frac{1}{2} \int d^3x \left\{ \left[\frac{1}{3} D_a R_{ibce} A_\mu^{ed} \partial^\mu \varphi^i + \frac{1}{3} R_{eabf} A_\mu^{ec} A^{\mu fd} + \frac{1}{12} D_a D_b R_{icdj} \partial_\mu \varphi^i \partial^\mu \varphi^j + \right. \right. \\
&\left. + \frac{1}{3} R^m_{abi} R_{mcdj} \partial_\mu \varphi^i \partial^\mu \varphi^j \right] \xi^a \xi^b \xi^c \xi^d + \frac{1}{2} D_a R_{ibcd} \partial_\mu \varphi^i \xi^a \xi^b \xi^c \partial^\mu \xi^d + \\
&\left. + \frac{1}{3} R_{cabe} A_\mu^{ed} \xi^a \xi^b \partial^\mu \xi^c \xi^d + R_{cabd} A_\mu^{ec} \xi^a \xi^b \xi^c \partial^\mu \xi^d + \frac{1}{3} R_{cabd} \xi^a \xi^b \partial_\mu \xi^c \partial^\mu \xi^d \right\}
\end{aligned}
\tag{4}
$$

4

where R is the curvature tensor and $A_\mu^{ab} \equiv \omega_i^{ab} \partial_\mu \varphi^i$ a vector potential $\in SO(n)$. We have also omitted in the expression (4),the linear term in ξ, $\mathcal{S}^{(1)}$, since it vanishes as we use the equation of motion of φ. Futhermore, we are not interested here in renormalization of wave function for which it could contribute. For computation up to two-loop order, the above results are all we need.

We are now able to compute the Feynman propagator for the ξ field, which is

$$\langle 0 \mid T\xi^a(x)\xi^b(y) \mid 0 \rangle = \delta^{ab}\Delta(x-y) \tag{5}$$

Before we start the calculation of the counterterms, a note is needed. We do not use dimensional regularization in this work, since in odd dimensional space-time it is in fact a renormalization prescription, which deletes all divergent contributions automatically, rendering the theory finite. Thus, if we wish to study the regularization effects in detail, we must make the counterterm structure explicit. We therefore choose a Pauli–Villars regularization (subtracting the infinites with the use of a regulator mass). Using this gauge invariant procedure, the second and third diagrams of Fig. 1 (solid and double lines denote the ξ and the background fields respectively) do not contribute and the divergent one-loop counterterm arises from the first one, which is given by

$$D^{(1)} = R_{iabj}\partial_\mu\varphi^i\partial^\mu\varphi^j\delta^{ab} \int \frac{d^3k}{(2\pi)^3} \frac{i}{(k^2 - \mu^2)} = \frac{\Lambda}{4\pi^2}C_{aa} \tag{6}$$

where $C_{aa} \equiv R_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j$, R_{ij} is the Ricci tensor and Λ a cutoff introduced after a Wick rotation. We have also included in the propagator (5) a mass μ to avoid infrared divergences, which are not the issue of this work.

According to Friedan's interpretation[8], this counterterm may be absorved in a redefinition of the metric as

$$g_{ij}^{ren} = g_{ij} - \frac{\Lambda}{4\pi^2}R_{ij} \,. \tag{7}$$

Thus we get an one-loop renormalized effective action

$$\mathcal{S}_{eff}^{(1)} = \frac{1}{2} \int d^3x \left(g_{ij} - \frac{\Lambda}{4\pi^2} C_{aa} \right). \tag{8}$$

Futhermore, we can define a β function as

$$\beta_{ij} \equiv \frac{\delta g_{ij}^{ren}}{\delta\Lambda} = -\frac{1}{4\pi^2} R_{ij}. \tag{9}$$

So we find that Ricci-flat manifolds are one-loop finite.

Now we shall perform the calculation of the two-loop counterterms. We have in this case several contributions and the diagrams are displayed in Fig. 2. However, we would like to appoint out that several vertices in the expansion(4) contain derivatives and as a consequence there will be a momentum flow through each vertice becoming such diagrams much more complicated, so an special attention will be paid in this calculation. Moreover, in order to define a renormalization procedure, we shall use here the BPHZ-type scheme making Taylor subtractions around zero external momenta[16]. The integrals appearing in this approach will evaluated, when necessary, by applying the Feynman parametrization method and using the Ref. [17].

For the first diagram, we have the square of an one-loop diagram, which is easily computable and the result is

$$\begin{aligned} D^{(2a)} &= \frac{\Lambda^2}{4\pi^4} \left[\frac{1}{4} [D_a R_{ib} - 3D_b R_{ia}] \omega_j^{ab} + \frac{1}{6} R_{ac} \omega_i^{ab} \omega_j^{bc} + \frac{1}{3} R_{c(ab)d} \omega_i^{cb} \omega_j^{da} + \frac{1}{12} D^a D_i R_{aj} - \right. \\ &\quad \left. - \frac{1}{8} D_a D^a R_{ij} + \frac{1}{4} R_{iabc} R_j^{abc} + \frac{1}{6} R_{ia} R_j^a \right] \partial_\mu \varphi^i \partial^\mu \varphi^j - \frac{\Lambda^4}{72\pi^4} R, \tag{10} \end{aligned}$$

where the symbol ( ) in tensor indices denotes symmetrization. Also, in the above expression, the Bianchi and cyclic indenties have been used. The contribution (10) implies still a redefinition of the vacuum. It is in essence a one-loop counterterm. Actually, althoug not in the usual sense, since the g_{ij} does not couple to a spatial derivative, it corresponds to a *cosmological term.*

6

The above procedure applied for diagrams (2b) and (2c) gives

$$
D^{(2b+2c)} = \frac{1}{192\pi^4}\left(\frac{\pi}{3}\frac{\Lambda^3}{\mu} + 4\Lambda^2\right)(R_{ab}\omega_i^{ca}\omega_j^{cb} + R^{ab}R_{iabj})\partial_\mu\varphi^i\partial^\mu\varphi^j +
$$
$$
+ \frac{1}{32\pi^3}\frac{\Lambda}{\mu}X_{ijpq}\partial_\mu\varphi^i\partial^\mu\varphi^j\partial_\nu\varphi^p\partial^\nu\varphi^q, \tag{11}
$$

where

$$
X_{ijpq} \equiv \frac{1}{4}\Big[(D_aR_{ic} + D_iR_{ac} - 2D_cR_{ia})R_{p(cb)q} + (D_bR_{idca} + D_bR_{idca}+
$$
$$
+ D_dR_{ibca})R_{p(dc)q}\Big]\omega_j^{ab} + \frac{1}{3}R_{a(bc)d}R_{p(be)q}(\omega_i^{ac}\omega_j^{de} + \omega_i^{ae}\omega_j^{dc}) - \frac{1}{2}R_{ab}R_{pcdq}\omega_i^{ac}\omega_j^{bd}
$$
$$
+ \frac{1}{2}R_{abcd}R_{pbcq}\omega_i^{ae}\omega_j^{de} + \frac{1}{48}\Big[2D^2R_{iabj}R_{pabq} + (4D_aD_iR_{jb} - 5D_aD_bR_{ij})R_{pabq}\Big] +
$$
$$
+ \frac{1}{3}\Big[R_{abci}\Big(R_{a(bd)j}R_{p(dc)q} + R_{a(ec)j}R_{p(be)q}\Big) - R_{ai}R_{abcj}R_{p(bc)q}\Big] \ .
$$

Next we are going to consider the contribution (2d) which is really new since it contains the first *non-renormalizable* counterterm. We divide it into two pieces(2d/1 and 2d/2). The first, analogous to the previous, is

$$
D^{(2d/1)} = \frac{1}{2(2\pi)^6}Y_{ijpq}\partial_\mu\varphi^i\partial^\mu\varphi^j\partial_\nu\varphi^p\partial^\nu\varphi^q\int d^3kd^3l\frac{1}{[(k+l+p)^2+\mu^2](k^2+\mu^2)(l^2+\mu^2)} \tag{12}
$$

where,

$$
Y_{ijpq} \equiv \frac{1}{18}\Big[(D_aR_{pbcq} + D_bR_{pacq} + D_cR_{pabq})(D_aR_{ibcj} + 4R_{iabd}\omega_j^{dc}) + 4(2D_aR_{p(bc)q}+
$$
$$
+ D_cR_{pabq})R_{i(ab)d}\omega_j^{dc} + 16R_{iabd}(R_{p(ab)e}\omega_j^{dc}\omega_q^{ec} + R_{p(ac)e}\omega_j^{dc}\omega_q^{eb} + R_{p(bc)i}\omega_j^{dc}\omega_q^{ea})\Big] \ .
$$
$$
\tag{13}
$$

The second one, also non-renormalizable, is given by the following expression in momentum space:

$$
D^{(2d/2)} = \frac{2}{9(2\pi)^6}R_{iabc}(R_j^{abc} + R_j^{bac})\partial^\mu\varphi^i\partial^\nu\varphi^j\int d^3kd^3l\frac{[2l_\mu l_\nu + l_\mu(k+p)_\nu]}{[(k+l+p)^2+\mu^2](k^2+\mu^2)(l^2+\mu^2)} \tag{14}
$$

leading to higher derivative counterterms, that is,

$$
\begin{aligned}
D^{(2d)} &= -\frac{1}{192\pi^4}\left\{\left[8\Lambda^2 - \frac{1}{5}\pi^2\left(ln\frac{\Lambda^2}{\mu^2} - \frac{8}{3\pi}\frac{\Lambda}{\mu} - \frac{4}{3\pi^2}\frac{\Lambda^2}{\mu^2}\right)p^2\right]\eta^{\mu\nu} + \right.\\
&+ \left. \frac{1}{5}\pi^2\left(3ln\frac{\Lambda^2}{\mu^2} - \frac{4}{3\pi}\frac{\Lambda}{\mu} - \frac{4}{3\pi^2}\frac{\Lambda^2}{\mu^2}\right)p^\mu p^\nu\right\} R_{iabc}R_j{}^{abc}\partial_\mu\varphi^i\partial_\nu\varphi^j - \\
&- \frac{1}{64\pi^2}ln\frac{\Lambda^2}{\mu^2}Y_{ijpq}\partial_\mu\varphi^i\partial^\mu\varphi^j\partial_\nu\varphi^p\partial^\nu\varphi^q .
\end{aligned}
\tag{15}
$$

We have computed explicitly the counterterms in the following cases:

(i) Ricci-flat spaces $(R_{ij} = 0)$;

(ii) Locally symmetric spaces, where $D_i R_{jklm} = 0$, which include the the $O(n)$ and CP^{n-1} models.

In the first case we have finiteness at one loop level, but nonrenormalizability at two loops; the counterterm, which is given by

$$
\begin{aligned}
D^{(2)}_{R-f} &= D^{(2d)} - \frac{\Lambda^2}{8\pi^4}\left(R_{dabc}\omega_i^{db}\omega_j^{ac} - \frac{1}{2}R_{iabc}R_j^{abc}\right)\partial_\mu\varphi^i\partial^\mu\varphi^j + \\
&+ \frac{1}{32\pi^3}\frac{\Lambda}{\mu}X_{ijpq}(R_{ab} = 0)\partial_\mu\varphi^i\partial^\mu\varphi^j\partial_\nu\varphi^p\partial^\nu\varphi^q
\end{aligned}
\tag{16}
$$

is not of the form of the original Lagrangian.

In the second case we have a renormalizability at the one-loop order, but some infinites still remain at the two-loop order. Specifically, in the $O(n)$ non-linear sigma model, where the metric, curvature and Ricci tensors are, respectively, given by

$$
g_{ij}(\varphi) = \delta_{ij} + \frac{\varphi_i\varphi_j}{1 - |\varphi|^2}, \quad R_{ijkl} = g_{ik}(\varphi)g_{jl}(\varphi) - g_{il}(\varphi)g_{jk}(\varphi), \quad R_{ij} = (n-2)g_{ij}
\tag{17}
$$

we have, at the one-loop level,

$$
\delta\mathcal{L}^{(1)}_{O(n)} = \frac{1}{4\pi^2}(n-2)\Lambda g_{ij}(\varphi)\partial_\mu\varphi^i\partial^\mu\varphi^j ,
\tag{18}
$$

and, at the two-loop order,

$$
\begin{aligned}
\delta\mathcal{L}^{(2)}_{O(n)} &= -\frac{1}{48\pi^4}\Bigg\{\Bigg[(3n-n^2+4)+\frac{\pi}{12}(n^2-5n+6)\frac{\Lambda^3}{\mu}\Bigg]\eta^{\mu\nu}+\frac{\pi^2}{10}(n-2)\times \\
&\quad \times \Bigg[(3ln\frac{\Lambda^2}{\mu^2}-\frac{4}{3\pi}\frac{\Lambda}{\mu})p^\mu p^\nu-(ln\frac{\Lambda^2}{\mu^2}-\frac{8}{3\pi}\frac{\Lambda}{\mu}-\frac{4}{3\pi^2}\frac{\Lambda^2}{\mu^2})p^2\eta^{\mu\nu}\Bigg]\Bigg\}g_{ij}\partial_\mu\varphi^i\partial_\nu\varphi^j + \\
&\quad + \Bigg\{\frac{1}{96\pi^3}\frac{\Lambda}{\mu}\Big[(n^2-3n-3)g_{ij}g_{pq}+(n+3)g_{ip}g_{jq}\Big]-\frac{(n+1)}{72\pi^2}ln\frac{\Lambda^2}{\mu^2}g_{ij}g_{pq}\Bigg\}\times \\
&\quad \times \partial_\mu\varphi^i\partial^\mu\varphi^j\partial_\nu\varphi^p\partial^\nu\varphi^q\ .
\end{aligned}
\tag{19}
$$

Therefore, we can easily see from (18) and (19) that the $O(n)$ model is renormalizable at one-loop order and obviously finite whether $n = 2$. Nevertheless, it remains non-renormalizable at the two-loop level since the quartic term in the field φ in (19) is not completely cancelled, even whether we consider the $O(2)$ case where some important cancellations are obtained.

In the next section we shall go on to consider the case of the fermions and write down the corresponding countertems again up to two-loop order.

3. Supersymmetric extension

We shall consider now the supersymmetric extension of the action (1), which reads

$$
S = \frac{1}{4i}\int d^3x d^2\theta g_{ij}(\Phi^k)\overline{D\Phi}^i D\Phi^j,
\tag{20}
$$

where $D\Phi^i$ is the supercovariant derivative of Φ^i , such that, $D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\slashed{\partial}\theta)_\alpha$; being θ a two-component Majorana anticommuting variable and Φ^i a scalar superfield whose expansion in terms of the component fields ϕ^i (the scalar fields), ψ^i (the Majorana spinor which are the fermionic partners) and F^i (the auxiliary fields), in the Majorana represention for the gamma matrices, is given by

$$
\Phi^i = \phi^i(x) + \bar{\theta}\psi^i(x) + \frac{1}{2}\bar{\theta}\theta F^i(x).
\tag{21}
$$

Substituting the above relations back into (20) , integrating over the Grassmann variable θ by means of the standard rules of Berezin integration and eliminating the auxiliary fields, we finally obtain the Lagrangian in terms of component fields

$$L = \frac{1}{2} \left[g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + i g_{ij}(\phi) \bar{\psi}^i \gamma^\mu D_\mu \psi^j + \frac{1}{6} R_{ijkl} (\bar{\psi}^i \psi^k)(\bar{\psi}^j \psi^l) \right] , \tag{22}$$

where

$$(D_\mu \psi)^j = \partial_\mu \psi^j + \Gamma^j_{kl} \partial_\mu \phi^k \psi^l . \tag{23}$$

The background field method works well as in the previous case[9]. We consider the Fermi fields ψ^i to be quantum fields, avoiding background quantum splitting for anticommuting variables. We obtain

$$g_{ij}(\phi) \bar{\psi}^i \slashed{D} \psi^j = \left(g_{ij}(\varphi) + \frac{1}{3} R_{iklj} \right) \xi^k \xi^l \bar{\psi}^i \slashed{D} \psi^j + \frac{1}{2} R_{ijkl} \partial_\mu \varphi^l \xi^k (\bar{\psi}^i \gamma^\mu \psi^j) . \tag{24}$$

We can now write all relevant objects in terms of tangent space variables, using

$$\xi^a = e^a_i \xi^i , \quad \psi^a = e^a_i \psi^i , \quad (D_\mu \psi)^a = \partial_\mu \psi^a + \omega^{ab}_i \partial_\mu \phi^i \psi^b . \tag{25}$$

Gathering together all relevant informations, we obtain

$$S = S^{(0)}[\psi] + S^{(1)}[\psi] + S^{(2)}[\psi] + S^{(3)}[\psi] + S^{(4)}[\psi] + \cdots \tag{26}$$

being

$$
\begin{aligned}
S^{(0)} &= \frac{1}{2} \int d^3x \bar{\psi}^a (i\gamma^\mu \partial_\mu - \mu) \psi^a , & S^{(1)} &= \frac{1}{2} \int d^3x i \bar{\psi}^a \gamma^\mu A^{ab}_\mu \psi^b , \\
S^{(2)} &= \frac{1}{6} \int d^3x R_{acdb} \xi^c \xi^d i \bar{\psi}^a \gamma^\mu D_\mu \psi^b , & S^{(3)} &= \frac{1}{4} \int d^3x R_{abci} \partial_\mu \varphi^i \xi^c i \bar{\psi}^a \gamma^\mu \psi , \\
S^{(4)} &= \frac{1}{12} \int d^3x R_{abcd} \bar{\psi}^a \psi^c \bar{\psi}^b \psi^d ,
\end{aligned}
\tag{27}
$$

where a mass has been introduced again in order that we obtain infrared finite results. Using the Pauli–Villars regulator, we get a vanishing result at one-loop(see Fig. 3, with the dashed lines denoting the fermion propagators). In this calculation, the infrared regulator and the

10

ultraviolet cutoff were taken to be equal to those ones of the bosonic case(μ and Λ respectively) in order not to introduce an explicit breaking of supersymmetry. In the following we shall use this same procedure. Now, at the two-loop order, we have the contributions shown in Fig. 4. In the first diagram we have a contribution arising from $S^{(2)}[\psi]$, given by

$$\delta S^{(4a)}[\psi] = -\frac{1}{6} \int d^3x R_{acdb} \langle T[\xi^c \xi^d \bar{\psi}^a \gamma^\mu (\partial_\mu \psi^b + iA_\mu^{bc}\psi^c)]\rangle \tag{28}$$

Upon contracting the ξ and the ψ fields, we obtain

$$\delta S^{(4a)} = \frac{\Lambda^4}{36\pi^4} \int d^3x R(x) \tag{29}$$

which is analogous to previous computations(see Eq. (10)). Nevertheless, due to a factor of 2, there is no cancellation between these terms. Note that diagrams (b) and (c) do not contribute. Finally, the last contribution is

$$\delta S^{(4d)} = -\frac{1}{32} \int d^3x d^3y R_{abcd}(x)\partial_\mu\varphi^d(x)R_{efgh}(y)\partial_\nu\varphi_\nu\varphi^h(y)\langle T[(\xi^c\bar{\psi}^a\gamma^\mu\psi^b)(x)(\xi^g\bar{\psi}^e\gamma^\nu\psi^f)(y)]\rangle \tag{30}$$

which, after a lengthy calculation, is given by

$$\begin{aligned}
\delta S^{(4d)} &= \frac{1}{192\pi^4}\left\{\left[\Lambda^2 - \frac{1}{5}\pi^2(3ln\frac{\Lambda^2}{\mu^2} - \frac{4}{\pi}\frac{\Lambda}{\mu} + \frac{8}{3\pi^2}\frac{\Lambda^2}{\mu^2})p^2\right]\eta^{\mu\nu} + \right. \\
&\quad + \left. \frac{2}{15}\pi^2\left[ln\frac{\Lambda^2}{\mu^2} + \frac{1}{4\pi^2}\right]p^\mu p^\nu\right\}\int d^3x R_{iabc}R_j^{abc}\partial_\mu\varphi^i\partial_\nu\varphi^j .
\end{aligned} \tag{31}$$

Therefore, even though we have a finiteness at the one-loop level the supersymmetric extension is not sufficient to remove completely all those non-renormalizable counterterms at two-loop or higher order. This result is crucially different from that one in two dimensions. In two space-time dimensions non-linear sigma models have no two- or three-loop terms in their β-function on any target manifold[8, 9].

4. Conclusions

In this paper we have calculated the divergences in the effective action of the three-dimensional non-linear sigma models and determined their one- and two-loop counterterms by the covariantized background field method. We have shown that at the one-loop order all the divergences may be absorbed in renormalizable counterterms. In fact, in some particular cases, when we consider the supersymmetric extension or symmetric and Ricci-flat manifolds, such models are even finite. On the other hand, at the two-loop level, we have found non-renormalizable counterterms in all cases considered so that the one-loop renormalizability and finiteness are completely lost. For instance, in the supersymmetric case, cancellation between bosons and fermions is not enough to render the model renormalizable . We think that the same continues to be true for higher supersymmetry(we have been working explicitly the case $N = 2$). Restrictions of the manifold may result in the fact some counterterms might be zero, but not all of them. However, we believe that these two-loop results, though negative in the sense that we do not find any sensible renormalizable theory in any simple case, should not be used to discard the models studied so far. Actually, it is possible to extract physical information out of the results obtained at the one-loop level which can be important in view of the many applications of sigma models, as we have already mentioned in the Introduction of this paper. By the way, the quantum gravity is a well known case of a non-renormalizable theory whose divergences, at the one-loop order, can be absorbed in the counterterms leading a meaningful theory[18].

Moreover, it seems also important for us to point out the different result one obtains from the perturbation theory used here, and other results based on the large n behaviour[12, 19], which define a renormalizable theory. Indeed, from $1/n$ perturbation of the CP^{n-1} model one learns that the model has two phases, one having a massive n-plet and a massless abelian gauge field, and another with a massless $(n-1)$-plet and a gauge field displaying no pole in

the propagator. In these sigma models, cancellation of divergences are a consequence of the definition of the auxiliary field propagator[20], and the identity shown in Fig. 5.

We should also make some remarks concerning general four-dimensional non-linear sigma models. Although already studied many years ago[15], it is not difficult to obtain the first few counterterms using the background field method. Indeed, the Lagrangian

$$L = g_{ij}\bar{\psi}^i \not{D}\psi^j + g_{ij}D_\mu\varphi^i D^\mu\varphi^j \tag{32}$$

has a background-quantum expansion given by

$$L = L_{cl}(\varphi^a, \psi^a) + R_{iabj}\left(\partial_\mu\varphi^i\partial^\mu\varphi^j + \frac{1}{3}\bar{\psi}^i \not{D}\psi^j\right)\xi^a\xi^b \tag{33}$$

with a gauge field $A_\mu^{ab} = \omega_i^{ab}\partial_\mu\varphi^i$. The diagram with two, three and four A_μ^{ab} legs cannot be made to vanish, and we need a counterterm $F_{\mu\nu}^2$, which is non-renormalizable already at the one-loop level.

Therefore, we are led to conjecture that, in all case considered, several of the infinites we found are fake infinites produced by perturbation theory, or else. As matter of fact, we believe that the theory may also have different phases(not those ones above mentioned) and that such infinites, as already noted by several authors in other contexts[21], may have nothing to do with the physical content of the models investigated. This calls for an explanation.

Finally, we would like to mention that more recently considerable discussion about non-renormalizable interactions has been done[22, 23, 24] and even certain approaches for the corresponding theories have been proposed[23, 24]. In particular, J. Gegelia et al.[24] have developed a method to extract physical information out of the series of non-renormalizable theories which coincides with the usual renormalization procedure(in terms of counterterms) for renormalizable ones. We hope that our calculations as well as these recent works can be useful for our understanding of this subject.

Acknowledgements

I wish to thank Professor Elcio Abdalla for collaboration in early stages of this work, helpful discussions and reading of the manuscript. I would also like to thank the Center for Theoretical Physics at MIT for hospitality. This work was partially supported by CAPES(Brazil).

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Figure Captions

Figure 1. One-loop order contributions.

Figure 2. Two-loop contributions.

Figure 3. Vanishing contribution upon use of gauge invariant regularization.

Figure 4. Two-loop contribution for the supersymmetric case.

Figure 5. Cancellation mechanism in the $1/n$ expansion.
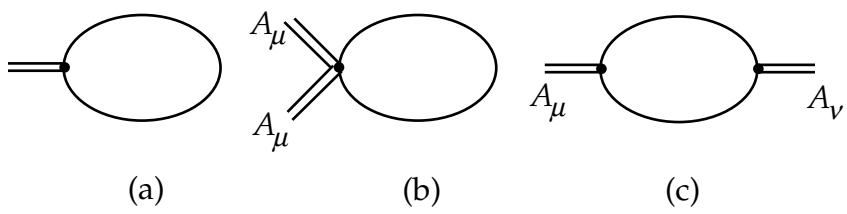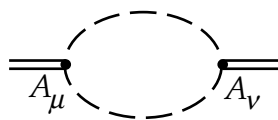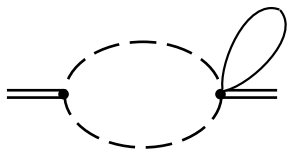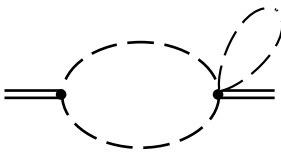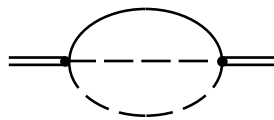
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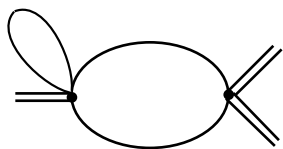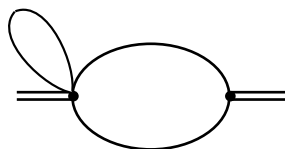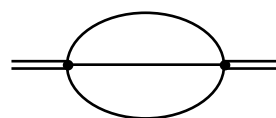
(a)   (b)   (c)   (d)

Figure 2

Figure 3

(a)   (b)   (c)   (d)

Figure 4

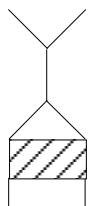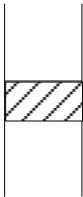Figure 5